Roll No. ....

(01/22-II)

5199

B.A./B.A. (Hons.)/B.Sc.

EXAMINATION

(Third Semester)

MATHEMATICS

BM-231

Advanced Calculus

Time : Three Hours *Maximum Marks* : B.Sc. : 40, B.A. : 27

Note : Attempt *Five* questions in all, selecting *one* question from each Section. Q. No. **1** is compulsory. All questions carry equal marks.

1. (i) Show that the function $f(x) = x^2$ is uniformly continuous in $[-2, 2]$.

(ii) If $u = f(x+2y) + g(x-2y)$, show that

$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}.$$

(iii) Give the statement of Young's theorem and Schwarz's theorem.

(iv) Find the equation of tangent line at the point $t = 1$ to the curve $x = 1+t$, $y = -t^2$, $z = 1+t^2$.

(v) Define curvature and torsion and hence give any *two* Serret-Frenet formulae.

Section I

2. (a) If a function f is uniformly continuous on $[a, b]$, then show that it is continuous on $[a, b]$. But the converse is not true, establish it with an example.

(b) Verify Roll's theorem for $f(x) = \sin 2x$ in $\left[0, \frac{\pi}{2}\right]$.

3. (a) If a function is twice differentiable on $[a, a+h]$, then show that :

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2} f''(a+\theta h),$$

$0 < \theta < 1$.

(b) Evaluate $\lim\limits_{x \to 0} \left(\frac{1}{x^2} - \cot^2 x \right)$.

Section II

4. (a) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by :

$$f(x, y) = \begin{cases} xy \dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & \text{otherwise} \end{cases}$$

is continuous at (0, 0).

(b) If $u = f(r)$, where $r = \sqrt{x^2 + y^2}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$.

5. (a) If $y^3 - 3ax^2 + x^3 = 0$, prove that :

$$\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0.$$

(b) If $u = \tan^{-1}\frac{x^3 + y^3}{x - y}$, prove that :

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 2u\left[1 - 4\sin^2 u\right].$$

Section III

6. Show that for the function :

$$f(x, y) = \begin{cases} \frac{x^2y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$f_{xy}(0,0) = f_{yx}(0,0)$ even though the conditions of Schwarz's theorem and also of Young's theorem are not satisfied.

7. (a) Examine for maximum and minimum values of the function :

$$\sin x + \sin y + \sin(x+y).$$

(b) Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.

Section IV

8. (a) Find the unit tangent vector $\hat{t}$ and the direction cosines of the tangent at any point to the circular helix $x = a\cos t$, $y = a\sin t$, $z = bt$.

(b) If the tangent and binormal at a point of a curve makes angle θ and ϕ with a fixed direction, show that $\frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi} = -\frac{x}{\tau}$.

9. (a) Show that the radius of spherical curvature of a circular helix $x = a\cos\theta$, $y = a\sin\theta$, $z = a\theta\cot\alpha$ is equal to the radius of circular curvature.

(b) Find the equations of tangent plane and normal to the surface $xyz = 4$ at the point (1, 2, 2).