Roll No. ....

(01/22-II)

5240

B. A./B.Sc. EXAMINATION

(Fifth Semester)

MATHEMATICS

BM-352

Groups and Rings

Time : Three Hours *Maxi. Marks :* B.Sc. : 40, B.A. : 26

Note : Attempt *Five* questions in all including compulsory question, selecting *one* question from each Section. Q. No. **1** is compulsory.

(Compulsory Question)

1. (a) Prove that the group G is abelian if and only if $(ab)^2 = a^2b^2$ for all a, b in G. 1(1)

(b) Let G be a group and H be a subgroup of G. Prove that $Ha = Hb$ if and only if $ab^{-1} \in H$. 2(1)

(c) Prove that the group of automorphisms of a finite cyclic group is abelian. 1(1)

(d) Give an example that a ring with unity element may have a subring not containing the unity element. 1(1)

(e) Find all the units of the ring $Z[i]$. 1(1)

(f) Show that the polynomial $x^4 + 1$ is irreducible over Q. 2(1)

Section I

2. (a) Let Q^+ be the set of all positive rational numbers and * be the binary operation defined by $a * b = \frac{ab}{3}$. Prove that Q^+ is an abelian group with respect to *. 4(2½)

(b) Prove that the order of a cyclic group is equal to the order of its generator. 4(2½)

3. (a) Prove that a sub-group H of a group G is normal in G if and only if each left coset of H in G is a right coset of H in G. 4(2½)

(b) Let {Z, +} be the group of all integers and $H = \{4n \mid n \in Z\}$. Find the right cosets of H in Z generated by 0, 1, 2, 3. Verify that Z is equal to the union of these cosets. 4(2½)

Section II

4. (a) Prove that every homomorphic image of a group G is isomorphic to some quotient group of G. 4(2½)

(b) Let G be a non-abelian group such that $o(G) = p^3$, where p is a prime. Show that order of the centre of group G is equal to p. 4(2½)

5. (a) Let G' be a commutator subgroup of G. Prove that G/G' is abelian and G' is the smallest subgroup of G such that G/G' is abelian. 4(2½)

(b) Find the centre of permutation group S_3, where $S = \{1, 2, 3\}$. 4(2½)

Section III

6. (a) Prove that $R = \{0, 1, 2, 3, 4, 5; +_6, \times_6\}$ is a ring. 4(2½)

(b) Prove that an ideal of a ring of integers is maximal if and only if it is generated by some prime integer. 4(2½)

7. (a) Prove that the order of each non-zero element of an integral domain (regarding the elements as the members of additive group) is same. 4(2½)

(b) Let $S \subseteq T$ be two subrings of a ring R. Prove that : 4(2½)

$$R/T \cong \frac{R/S}{T/S}.$$

Section IV

8. (a) Show that the integral domain $< Z, +, >$ of integers is a Euclidean domain. 4(2½)

   (b) Prove that every irreducible element in a principal ideal domain is aprime element. 4(2½)

9. (a) Let R be a UFD. Prove that product of to primitive polynomials in $R[x]$ is again a primitive polynomial. 4(2½)

   (b) If R is an integral domain with unity, then show that $R[x]$ is also an integral domain. 4(2½)