Roll No. ....

(01/22-II)

5239

B. A./B.Sc. EXAMINATION

(Fifth Semester)

MATHEMATICS

BM-351

Real Analysis

Time : Three Hours *Maxi. Marks :* B.Sc. : 40, B.A. : 27

Note : Attempt *Five* questions in all, selecting *one* question from each Section. Q. No. **1** is compulsory. All questions carry equal marks. Marks for B. A. will be 2/3 of total marks.

Section I

1. (a) A function f is defined on $[1, 3]$ as

$$f(x)=\begin{cases} x^2-\dfrac{|x|}{x} & \text{if } 1\le x\le 2 \\ 1 & \text{if } 2<x\le 3 \end{cases}$$

Is f integrable on [1, 3] ? If so, evaluate

$\int_1^3 f\,dx$. 2

(b) Examine the convergence of the integral

$\int_0^\infty \frac{dx}{1+x^2}$. 1

(c) Prove that every closed subset of a compact metric space is compact. 1

(d) If $a>0$, $b>0$, prove that : 1

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx=\log\frac{b}{a}$$

(e) Show that every totally bounded metric space is bounded. 1

(f) State finite intersection property in a metric space. Give an example also. 2

Section II

2. (a) Show that a bounded function having a finite number of points of discontinuity on $[a, b]$ is integrable on $[a, b]$. 4

(b) Evaluate $\int_1^3 (x^2+2x+3)dx$ by using limit of Riemann sums. 4

3. (a) Prove the inequality : 4

$$\frac{\sqrt{3}}{8} \le \int_{\pi/4}^{\pi/3} \frac{\sin x}{x} dx \le \frac{\sqrt{2}}{6}$$

(b) If f and g are integrable on $[a, b]$ and g keeps the same sign over $[a, b]$, then there exists a number λ between the bounds of f on $[a, b]$ such that : 4

$$\int_a^b fg\, dx = \lambda \int_a^b g\, dx.$$

Section III

4. (a) If f and g are two positive functions on $[a, b]$, a being the only point of infinite discontinuity such that :

$$\bigcup_{x \to a^+} \frac{f(x)}{g(x)} = l(\neq 0, \infty),$$ then the two integrals $\int_a^b f\,dx$ and $\int_a^b g\,dx$ converge or diverge together at 'a'. 4

(b) Show that $\int_1^\infty \frac{\sin x}{x}\,dx$ is convergent. 4.

5. (a) Show that the integral $\int_0^\infty x^{n-1} e^{-x}\,dx$ is convergent if $n > 0$. 4

(b) Prove that : 4

$$\int_0^{\pi/2} \frac{dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} = \frac{\pi(a^2 + b^2)}{4a^3 b^3}.$$

Section IV

6. (a) Prove that any metric space (X, d), bounded or not can be converted into a bounded metric space (X, d) where : 4

$$d^*(x,y) = \frac{d(x,y)}{1+d(x,y)}.$$

(b) Prove that in a metric space, the intersection of finite number of open sets is open. 4

7. (a) Prove that a subspace Y of a complete metric space X is complete iff it is closed. 4

(b) Show that in a metric space [0, 1] with usual metric $d(x,y) = |x-y|$, the sequence $a_n = \frac{1}{n}$ is a Cauchy sequence but does not converge in (0, 1]. 4

Section V

8. (a) Prove that the function $f : R \to R$ such that $f(x) = x^2$ for all $x \in R$ is not uniformly continuous on R. 4

   (b) Prove that a compact subset of a metric space is closed and bounded. 4

9. (a) Prove that continuous image of a connected space is connected. 4

   (b) Prove that a metric space (X, *d*) is compact iff every infinite subset A of X has a limit point in X (BWP). 4